DISCRETE MATHEMATICS

Restrained domination in trees

Gayla S. Domke[a], Johannes H. Hattingh[a,*], Michael A. Henning[b,1], Lisa R. Markus[c]

[a] *Department of Mathematics and Computer Science, Georgia State University, University Plaza, Atlanta, GA 30303-3083, USA*
[b] *Department of Mathematics and Applied Mathematics, University of Natal, Pietermaritzburg 3209, South Africa*
[c] *Department of Mathematics, Furman University, Greenville, SC. 29613, USA*

Received 20 October 1997; revised 9 September 1998; accepted 22 December 1998

Abstract

Let $G=(V,E)$ be a graph. A set $S\subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V-S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. We show that if T is a tree of order n, then $\gamma_r(T)\geqslant\lceil(n+2)/3\rceil$. Moreover, we constructively characterize the extremal trees T of order n achieving this lower bound. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

In this paper, we follow the notation of [1]. Specifically, let $G=(V,E)$ be a graph with vertex set V and edge set E. Moreover, the notation P_n will denote the path of order n. A set $S\subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [5,6].

In this paper, we continue the study of a variation of the domination theme, namely that of restrained domination [2–4,7]. A set $S\subseteq V$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V-S$. Every graph has a restrained dominating set, since $S=V$ is such a set. The *restrained domination*

* Corresponding author.
E-mail address: jhattingh@gsu.edu (J.H. Hattingh)

[1] Research supported in part by the University of Natal and the South African Foundation for Research Development.

PII: S0012-365X(99)00036-9

number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G.

The concept of restrained domination was introduced by Telle and Proskurowski [8], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set S, the complementary set $V-S$ and on edges between the sets S and $V-S$. For example, if we require that every vertex in $V-S$ should be adjacent to some other vertex of $V-S$ (the condition on the set $V-S$) and to some vertex in S (the condition on edges between the sets S and $V-S$), then S is a restrained dominating set.

One application of domination is that of prisoners and guards. For security, each prisoner must be seen by some guard; the concept is that of domination. However, in order to protect the rights of prisoners, we may also require that each prisoner is seen by another prisoner; the concept is that of restrained domination.

We show that if T is a tree of order n, then $\gamma_r(T) \geqslant \lceil (n+2)/3 \rceil$. Moreover, we constructively characterize the extremal trees T of order n achieving this lower bound. We refer to a vertex of degree 1 in T as a *leaf* of T. A vertex adjacent to a leaf we call a *remote vertex* of T. If u and v are adjacent vertices in T, then we shall denote the component of $T-uv$ containing u by $T(u,uv)$. For a vertex v of T, we shall use the expression, *attach a* P_m at v, to refer to the operation of taking the union of T and a path P_m and joining one of the ends of this path to v with an edge.

2. The lower bound

In [2], it was shown that if $n \geqslant 1$, then $\gamma_r(P_n) \geqslant \lceil (n+2)/3 \rceil$. As we shall see, this result will be useful in establishing a sharp lower bound on the restrained domination number of a tree.

Theorem 1. *If T is a tree of order $n \geqslant 1$, then $\gamma_r(T) \geqslant \lceil (n+2)/3 \rceil$.*

Proof. We use induction over n. It is easy to check that the result is true for all trees of order $n \leqslant 5$. Suppose, therefore, that the result is true for all trees of order less than n, where $n \geqslant 6$. Let $\gamma_r = \min\{\gamma_r(T) \mid T$ is a tree of order $n\}$. We will show that $\gamma_r \geqslant \lceil (n+2)/3 \rceil$.

Let $\mathscr{T} = \{T \mid T$ is a tree of order n such that $\gamma_r(T) = \gamma_r\}$. Among all trees in $\mathscr{T}$, let T be chosen so that the sum $s(T)$ of the degrees of its vertices of degree at least 3 is a minimum. If $s(T)=0$, then $T \cong P_n$, and so $\gamma_r = \gamma_r(P_n) \geqslant \lceil (n+2)/3 \rceil$. Suppose, therefore, that $s(T) \geqslant 1$. Let S be a minimum restrained dominating set of T. $\square$

Fact 1. *If* $\deg v \geqslant 3$, *then*

(i) $v \notin S$,

(ii) v *is adjacent to exactly one vertex of* S,

(iii) $\deg v = 3$.

Proof. Let $N(v) = \{v_1, v_2, v_3, \ldots, v_d\}$. Consider the tree T to be rooted at v. For $i = 1, 2, \ldots, d$, let T_i be the subtree of T induced by v_i and its descendants, and let ℓ_i be a leaf of T which is also in T_i. If $v \in S$, then, since every leaf of T belongs to S, $\ell_1 \in S$. The set S is also a restrained dominating set of the tree T' obtained from T by deleting the edge vv_2 and adding the edge $v_2\ell_1$, and so $T' \in \mathscr{T}$. However, $s(T') < s(T)$, contradicting our choice of S. Hence $v \notin S$. Thus v is adjacent to a vertex in S, v_1 say, and to a vertex not in S, v_2 say. If $v_k \in S$ for some k, $3 \leqslant k \leqslant d$, then S is a restrained dominating set of the tree T' obtained from T by deleting the edge vv_k and adding the edge $v_k\ell_1$. Thus $T' \in \mathscr{T}$. However, $s(T') < s(T)$, contradicting our choice of S. Hence v_1 is the only neighbor of v in S. If $d \geqslant 4$, then S is a restrained dominating set of the tree T' obtained from T by deleting the edges vv_3 and vv_4 and adding the edges v_3v_4 and $v_3\ell_1$. Thus $T' \in \mathscr{T}$. However, $s(T') < s(T)$, contradicting our choice of S. Hence $d = 3$. □

Fact 2. *No two vertices of degree* 3 *are adjacent.*

Proof. Using the notation employed in Fact 1, v_1 is the only neighbor of v in S. By Fact 1, $\deg v_1 \leqslant 2$. If $\deg v_2 = 3$, then, by Fact 1, v_2 is adjacent to a vertex of T_2 not in S and to a vertex of T_2 in S. Then S is a restrained dominating set of the tree T' obtained from T by deleting the edge vv_2 and adding the edge $\ell_1 v_2$. Thus $T' \in \mathscr{T}$. However, $s(T') < s(T)$, contradicting our choice of S. Hence $\deg v_2 \leqslant 2$. Since v_2 is adjacent to a vertex of S, it follows that $\deg v_2 = 2$. Similarly, $\deg v_3 = 2$. □

Using the notation employed in Fact 1, if v_1 is not a leaf, then, by Fact 1, $\deg v_1 = 2$. Let v_1' be the neighbor of v_1 different from v. Then S is a restrained dominating set of the tree T' obtained from T by deleting the edge v_1v_1' and adding the edge $v_1'\ell_2$. Thus $T' \in \mathscr{T}$ and v_1 is a leaf of T'. Hence we may assume that v_1 is a leaf of T. For $i = 2, 3$, let v_i' be the neighbor of v_i different from v. Necessarily, $v_2', v_3' \in S$. We may assume, as we did for v_1, that v_2' is a leaf of T.

If $n = 6$, then $\gamma_r(T) = 3 = \lceil (n+2)/3 \rceil$. Suppose, therefore, that $n \geqslant 7$. Let T' be the subtree of T induced by v_3' and its descendants. Then $S \cap V(T')$ is a restrained dominating set of T', so that $|S \cap V(T')| \geqslant \gamma_r(T')$. Hence, $|S| \geqslant 2 + \gamma_r(T')$. Applying the inductive hypothesis to the tree T' of order $n-5$, we have $\gamma_r(T') \geqslant \lceil (n-3)/3 \rceil$, and so $\gamma_r(T) = |S| \geqslant \lceil (n+3)/3 \rceil \geqslant \lceil (n+2)/3 \rceil$. □

3. The characterization

For $n \geqslant 1$, let $\mathscr{T}_n = \{T \mid T \text{ is a tree of order } n \text{ such that } \gamma_r(T) = \lceil (n+2)/3 \rceil\}$. We will present a constructive characterization of the family $\mathscr{T}$. For this purpose, we define a type (1) operation on a tree T as attaching a P_2 at v where v is a vertex of T not belonging to some minimum restrained dominating set of T, and a type (2) operation as attaching a P_3 at v where v belongs to some minimum restrained dominating set of T. For $i=1,2$, let T_i be the tree obtained from $K(1,3)$ by subdividing i edges once.

We now define three families of trees as follow. Let $\mathscr{C}_{3k}=\{T \mid T$ is a tree of order $3k$ which can be obtained from the tree T_2 by a finite sequence of operations of type $(2)\}$. Let $\mathscr{C}_{3k+1}=\{T \mid T$ is a tree of order $3k+1$ which can be obtained from P_4 by a finite sequence of operations of type $(2)\}$. Finally, let $\mathscr{C}_{3k+2}=\{T \mid T$ is a tree of order $3k+2$ which can be obtained from P_5 or from the tree T_1 by a finite sequence of operations of type $(2)\} \cup \{T \mid T$ is a tree of order $3k+2$ which can be constructed from the tree T_2 by a finite sequence of operations of type (2), followed by one operation of type (1) and then by a finite sequence of operations of type $(2)\}$.

We shall establish:

Theorem 2. *For $n \geqslant 4$, $\mathscr{T}_n=\mathscr{C}_n$.*

We note the following simple, but frequently used, fact.

Lemma 3. *If D is a minimum restrained dominating set of a tree T, then every leaf of T belongs to D.*

We prove Theorem 2 by establishing eight lemmas.

Lemma 4. *If $T \in \mathscr{T}_n$, then each vertex of T is adjacent to at most two leaves. Furthermore, at most one vertex of T is adjacent to two leaves.*

Proof. Let D be a minimum restrained dominating set of T. Since $T \in \mathscr{T}_n$, $|D| = \lceil (n+2)/3 \rceil$. Suppose that some vertex v of T is adjacent to at least three leaves, say v_1, v_2 and v_3. Then $D-\{v_1,v_2\}$ is a restrained dominating set of $T'=T-\{v_1,v_2\}$. Hence $\gamma_r(T') \leqslant |D|-2=\lceil (n-4)/3 \rceil$. However, T' is a tree of order $n-2$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil n/3 \rceil$. Consequently, $\lceil n/3 \rceil \leqslant \lceil (n-4)/3 \rceil$, which is impossible. Hence no vertex of T can be adjacent to more than two leaves. Furthermore, suppose that u and v are distinct vertices of T that are both adjacent to two leaves. Let ℓ_1 and ℓ_2 be the two leaves adjacent to u, and let ℓ_3 and ℓ_4 be the two leaves adjacent to v. Then $D-\{\ell_1,\ell_3\}$ is a restrained dominating set of $T'=T-\{\ell_1,\ell_3\}$. Hence $\gamma_r(T') \leqslant |D|-2=\lceil (n-4)/3 \rceil$. However, $\gamma_r(T') \geqslant \lceil n/3 \rceil$, and once again we get a contradiction. Hence at most one vertex of T is adjacent to two leaves. $\square$

Lemma 5. *If $T \in \mathscr{T}_n$ and $n \not\equiv 2 \pmod 3$, then each vertex of T is adjacent to at most one leaf, and no remote vertex of T belongs to a minimum restrained dominating set of T.*

Proof. Let D be a minimum restrained dominating set of T. Suppose that some vertex v of T is adjacent to two leaves, say ℓ_1 and ℓ_2. Then $D-\{\ell_1\}$ is a restrained dominating set of $T'=T-\ell_1$. Hence, since $T \in \mathscr{T}_n$, $\gamma_r(T')=\gamma_r(T)-1=\lceil (n-1)/3 \rceil$. However,

T' is a tree of order $n-1$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil (n+1)/3 \rceil$. Consequently, $\lceil (n-1)/3 \rceil \geqslant \lceil (n+1)/3 \rceil$, which is impossible unless $n \equiv 2 \pmod 3$. Hence no vertex of T is adjacent to more than one leaf. Furthermore, if v is adjacent to a leaf ℓ_1, and if $v \in D$, then once again $D - \{\ell_1\}$ is a restrained dominating set of $T' = T - \ell_1$, which is a contradiction unless $n \equiv 2 \pmod 3$. $\square$

Lemma 6. *If* $T \in \mathscr{T}_n$ *and* T' *is obtained from* T *by a type* (2) *operation, then* $T' \in \mathscr{T}_{n+3}$.

Proof. Let D be a minimum restrained dominating set of T. Then adding the new leaf of T' to D produces a restrained dominating set of T'. Hence, since $T \in \mathscr{T}_n, \gamma_r, (T') \leqslant \gamma_r(T) + 1 = \lceil (n+5)/3 \rceil$. However, T' is a tree of order $n+3$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil (n+5)/3 \rceil$. Consequently, $\gamma_r(T') = \lceil (n+5)/3 \rceil$, and so $T' \in \mathscr{T}_{n+3}$. $\square$

Since $\mathscr{C}_4 = \{P_4\} = \mathscr{T}_4$ and $\mathscr{C}_6 = \{T_2\} \subseteq \mathscr{T}_6$, we have $\mathscr{C}_k \subseteq \mathscr{T}_k$ for $k=4$ and $k=6$. Hence an immediate consequence of Lemma 6 now follows.

Lemma 7. *If* $n \geqslant 4$ and $n \not\equiv 2 \pmod 3$, *then* $\mathscr{C}_n \subseteq \mathscr{T}_n$.

Lemma 8. *If* $n \geqslant 4$ *and* $n \not\equiv 2 \pmod 3$, *then* $\mathscr{T}_n \subseteq \mathscr{C}_n$.

Proof. We proceed by induction on $n \geqslant 4$. Since $\mathscr{T}_4 = \{P_4\} = \mathscr{C}_4$ and $\mathscr{T}_6 = \{T_2\} = \mathscr{C}_6$, the result is true for $n=4$ and 6. Let $n \geqslant 7$ satisfy $n \not\equiv 2 \pmod 3$, and suppose that $\mathscr{T}_k \subseteq \mathscr{C}_k$ for all integers $k \not\equiv 2 \pmod 3$, where $4 \leqslant k < n$. Let $T \in \mathscr{T}_n$. We show that $T \in \mathscr{C}_n$. Let D be a minimum restrained dominating set of T. Let $P: v_1, v_2, \ldots, v_m$ be a longest path in T. Then v_1 is a leaf, and so $v_1 \in D$. Since P is a longest path, it follows from Lemma 5 that v_2 is a remote vertex of degrees 2 and $v_2 \notin D$. Hence $v_3 \notin D$. We show that $\deg v_3 = 2$ or $\deg v_{m-2} = 2$. If this is not the case, then $\deg v_3 \geqslant 3$ and $\deg v_{m-2} \geqslant 3$. Let $T^* = T(v_3, v_3v_4)$. Before proceeding further, we prove the following facts.

Fact 3. *If* v_3 *is not a remote vertex, then* $T^* \cong P_5, v_4 \in D$, *and* $D \cap V(T^*)$ *consists of the end-vertices of* T^*.

Proof. Since $\deg v_3 \geqslant 3$ and P is a longest path. T^* can be obtained from a star $K(1, r+1), r \geqslant 1$, with center v_3 by subdividing each edge exactly once. Each leaf of T^* belongs to D, while, by Lemma 5, D, contains no remote vertex of T^*. Since $v_3 \notin D$, $v_4 \in D$. Hence, if T' denotes the tree obtained from T by removing all vertices of T^* different from $\{v_1, v_2, v_3\}$, then $\gamma_r(T') \leqslant |D| - r$. Thus, since $T \in \mathscr{T}_n, \gamma_r(T') \leqslant \lceil (n+2-3r)/3 \rceil$. However, T' is a tree of order $n-2r$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil (n+2-2r)/3 \rceil$. Consequently, $\lceil (n+2-2r)/3 \rceil \leqslant \lceil (n+2-3r)/3 \rceil$. Since $n \not\equiv 2 \pmod 3$, this is impossible unless $r=1$ and $n \equiv 0 \pmod 3$. $\square$

Fact 4. *If v_3 is a remote vertex, then $T^* \cong P_4$, $v_4 \notin D$, and $D \cap V(T^*)$ consists of the end-vertices of T^*.*

Proof. By Lemma 5, v_3 is adjacent to only one leaf, say to ℓ_1. If $v_4 \in D$, then $D - \{\ell_1\}$ is a restrained dominating set of $T' = T - \ell_1$. Thus, since $T \in \mathscr{T}_n, \gamma_r(T') \leqslant \lceil (n-1)/3 \rceil$ However, T' is a tree of order $n-1$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil (n+1)/3 \rceil$. Consequently, $\lceil (n+1)/3 \rceil \leqslant \lceil (n-1)/3 \rceil$, which is impossible since $n \not\equiv 2 \pmod 3$. Hence $v_4 \notin D$. The tree T^* can be obtained from a star $K(1, r+1), r \geqslant 1$, with center v_3 by subdivding r edges exactly once. Each leaf of T^* belongs to D, while, by Lemma 5, D contains no remote vertex of T^*. Hence, if T' denotes the tree obtained from T by removing all vertices of T^* different from $\{v_3, \ell_1\}$, then $\gamma_r(T') \leqslant |D| - r = \lceil (n+2-3r)/3 \rceil$. However, T' is a tree of order $n - 2r$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil (n+2-2r)/3 \rceil$. Consequently, $\lceil (n+2-2r)/3 \rceil \leqslant \lceil (n+2-3r)/3 \rceil$, which is impossible unless $r = 1$ and $n \equiv 0 \pmod 3$. □

Similar statements to Facts 3 and 4 hold for the tree $T(v_{m-2}, v_{m-2}, v_{m-3})$. Hence, if T' denotes the tree obtained from T by removing the vertices v_1, v_2, v_{m-1}, v_m, then $D - \{v_1, v_m\}$ is a restrained dominating set of T', and so $\gamma_r(T') \leqslant |D| - 2 = \lceil (n-4)/3 \rceil$. However, T' is a tree of order $n - 4$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil (n-2)/3 \rceil$. Consequently, $\lceil (n-2)/3 \rceil \leqslant \lceil (n-4)/3 \rceil$, which is impossible since $n \not\equiv 2 \pmod 3$. We deduce, therefore, that $\deg v_3 = 2$ or $\deg v_{m-2} = 2$.

We may assume that $\deg v_3 = 2$. Then $v_4 \in D$. Hence $D - \{v_1\}$ is a restrained dominating set of $T' = T - \{v_1, v_2, v_3\}$. Thus, since $T \in \mathscr{T}_n, \gamma_r(T') \leqslant |D| - 1 = \lceil (n-1)/3 \rceil$. However, T' is a tree of order $n-3$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil (n-1)/3 \rceil$. Consequently, $\gamma_r(T') = \lceil (n-1)/3 \rceil$. Hence $T' \in \mathscr{T}_{n-3}$. By the inductive hypothesis, $\mathscr{T}_{n-3} \subseteq \mathscr{C}_{n-3}$, and so $T' \in \mathscr{C}_{n-3}$. However, T is constructed from T' by a type (2) operation. Hence, by Lemma 6, $T \in \mathscr{C}_n$. □

Lemma 9. *If $T \in \mathscr{T}_n$ and $n \not\equiv 2 \pmod 3$, then the tree T' obtained from T by a type* (1) *operation belongs to $\mathscr{T}_{n+2}$.*

Proof. Let D be a minimum restrained dominating set of T. Then adding the new leaf of T' to D produces a restrained dominating set of T'. Hence, since $T \in \mathscr{T}_n$ and $n \not\equiv 2 \pmod 3$, $\gamma_r(T') \leqslant \gamma_r(T) + 1 = \lceil (n+5)/3 \rceil = \lceil (n+4)/3 \rceil$. However, T' is a tree of order $n+2$, and so, by Theorem 1, $\gamma_r(T') \geqslant \lceil (n+4)/3 \rceil$. Consequently, $\gamma_r(T') \geqslant \lceil (n+4)/3 \rceil$, and so $T' \in \mathscr{T}_{n+2}$. □

Lemma 10. *If $n \geqslant 1$, then $\mathscr{C}_{3n+2} \subseteq \mathscr{T}_{3n+2}$.*

Proof. We proceed by induction on $n \geqslant 1$. The base case is true since $\mathscr{C}_5 = \{P_5, T_1\} \subseteq \mathscr{T}_5$. For $n \geqslant 1$, suppose that $\mathscr{C}_{3n+2} \subseteq \mathscr{T}_{3n+2}$. We show that $\mathscr{C}_{3(n+1)+2} \subseteq \mathscr{T}_{3(n+1)+2}$. Let $\mathscr{T} \in \mathscr{C}_{3(n+1)+2}$. Then T can be obtained from a tree T' by either one operation of type (1) or one operation of type (2). If T is obtained from T' by one operation of type (1), then T' has order $3n+3$ and the construction of T' is accomplished by using only type (2) operations starting with the tree T_2. Hence, by Lemma 6, $T' \in \mathscr{T}_{3n+3}$. Thus, by Lemma 9, $T \in \mathscr{T}_{3(n+1)+2}$. On the other hand, if T is obtained from T' by one operation of type (2), then T' has order $3n+2$ and $T' \in \mathscr{C}_{3n+2}$. By our induction hypothesis, $T' \in \mathscr{T}_{3n+2}$, and, by Lemma 6, $T \in \mathscr{T}_{3n+5}$. $\square$

Lemma 11. *If* $n \geqslant 1$, *then* $\mathscr{T}_{3n+2} \subseteq \mathscr{C}_{3n+2}$.

Proof. We proceed by induction on $n \geqslant 1$. Since $\mathscr{T}_5 = \{P_5, T_1\} = \mathscr{C}_5$, the result is true for $n=1$. For $n \geqslant 2$ suppose that $\mathscr{T}_{3k+2} \subseteq \mathscr{C}_{3k+2}$ for all integers k, where $1 \leqslant k < n$. Let $T \in \mathscr{T}_{3n+2}$. We show that $T \in \mathscr{C}_{3n+2}$. Let D be a minimum restrained dominating set of T. By Theorem 1, $|D| = n+2$. Let $P: v_1, v_2, \ldots, v_m$ be a longest path in T. Then v_1, v_m are leaves, and so $v_1, v_m \in D$, By Lemma 4, we may assume that v_2 is a remote vertex of degree 2. We consider two possibilities, depending on whether v_{m-1} is adjacent to one leaf or to two leaves.

Let $T^* = T(v_3, v_3v_4)$.

Case 1: $\deg v_{m-1} = 3$. Then v_{m-1} is adjacent to two leaves. If $v_2 \in D$, then $D - \{v_1, v_m\}$ is a restrained dominating set of $T' = T - \{v_1, v_m\}$. But then $\gamma_r(T') \leqslant |D| - 2 = n$ and $\gamma_r(T') \geqslant \lceil (3n+2)/3 \rceil = n+1$, which is impossible. Hence $v_2 \notin D$, and therefore $v_3 \notin D$.

Fact 5. $\deg v_3 = 2$.

Proof. Suppose that $\deg v_3 \geqslant 3$. Suppose that v_3 is not a remote vertex. Then, since P is a longest path, T^* can be obtained from a star $K(1, r+1), r \geqslant 1$, with center v_3 by subdividing each edge exactly once. Furthermore, by Lemma 4, each remote vertex of T^* has degree 2 and, as shown with v_2, D contains no remote vertex of T^*. Since $v_3 \notin D$, $v_4 \in D$. Hence, if T' denotes the tree obtained from T by removing all vertices of T^* different from $\{v_1, v_2, v_3\}$ and by removing v_m, then $\gamma_r(T') \leqslant |D| - r - 1 = n + 1 - r$. However, T' is a tree of order $3n+1-2r$, and so, by Theorem 1, $\gamma_r(T') \geqslant n+1-\lfloor 2r/3 \rfloor$, which is a contradiction. Hence v_3 is a remote vertex.

By Lemma 4, v_3 is adjacent to only one leaf, say to ℓ_1. If $v_4 \in D$, then $D - \{\ell, v_m\}$ is a restrained dominating set of $T' = T - \{\ell_1, v_m\}$, which as before is a contradiction. Hence $v_4 \notin D$. The tree T^* can be obtained from a star $K(1, r+1), r \geqslant 1$, with center v_3 by subdividing r edges exactly once. Each leaf of T^* belongs to D, while D contains no remote vertex of T^*. Hence, if T' denotes the tree obtained from T by removing all vertices of T^* different from $\{v_3, \ell_1\}$ and by removing v_m, then $\gamma_r(T') \leqslant |D| - r - 1 = n + 1 - r$. However, $\gamma_r(T') \geqslant n + 1 - \lfloor 2r/3 \rfloor$, which is a contradiction. Hence $\deg v_3 = 2$. $\square$

By Fact 5, $\deg v_3 = 2$. Thus $v_4 \in D$. Hence $D - \{v_1\}$ is a restrained dominating set of $T' = T - \{v_1, v_2, v_3\}$, and so $\gamma_r(T') \leqslant |D| - 1 = n+1$. However, T' is a tree of order $3n-1$, and so, by Theorem 1, $\gamma_r(T') \geqslant n+1$. Consequently, $\gamma_r(T') = n+1 = \lceil (3n+1)/3 \rceil$. Hence $T' \in \mathscr{T}_{3(n-1)+2}$. By the inductive hypothesis, $\mathscr{T}_{3(n-1)+2} \subseteq \mathscr{C}_{3(n-1)+2}$, and so $T' \in \mathscr{C}_{3(n-1)+2}$. However, T is constructed from T' by a type (2) operation. Hence, by Lemma 6, $T \in \mathscr{C}_{3n+2}$.

Case 2: $\deg v_{m-1}=2$. If both v_2 and v_{m-1} belong to D, then $D-\{v_1, v_m\}$ is a restrained dominating set of $T' = T - \{v_1, v_m\}$, producing a contradiction. Hence, we may assume that $v_2 \notin D$. Thus $v_3 \notin D$. If $v_4 \notin D$, then $D - \{v_1\}$ is a restrained dominating set of $T' = T - \{v_1, v_2\}$. Hence $\gamma_r(T') \geqslant n+1$. However, T' is a tree of order $3n$, and so, by Theorem 1, $\gamma_r(T') \geqslant n+1$. Consequently, $\gamma_r(T') = n+1$, and so $T' \in \mathscr{T}_{3n}$. Thus, by Lemma 8, $T' \in \mathscr{C}_{3n}$. Since T can be constructed from T' by a type (1) operation, it follows from Lemma 9, that $T \in \mathscr{C}_{3n+2}$. So we may assume that $v_4 \in D$. Furthermore, if $\deg v_3 = 2$, then, as shown in Case 1. $T \in \mathscr{C}_{3n+2}$. So we may assume that $\deg v_3 \geqslant 3$.

Fact 6. *If v_3 is not a remote vertex, then $T \in \mathscr{C}_{3n+2}$.*

Proof. Let u be a neighbor of v_3 distinct from v_2 and v_4. Suppose $u \in D$. Now u is adjacent to either one leaf or to two leaves. In any event, the tree T' obtained from T by deleting $N[u] - \{v_3\}$ satisfies $\gamma_r(T') \leqslant |D| - \deg u \leqslant n$. However, T' has order at least $3n-1$, and so, by Theorem 1, $\gamma_r(T') \geqslant n+1$, producing a contradiction. Hence $u \notin D$. But then $D - \{v_1\}$ is a restrained dominating set of $T' = T - \{v_1, v_2\}$, whence, as shown earlier, $T' \in \mathscr{T}_{3n}$ and $T \in \mathscr{C}_{3n+2}$. $\square$

By Fact 6, we may assume that v_3 is a remote vertex of T. If v_3 is adjacent to two leaves, then removing these two leaves from T produces a tree T' with $\gamma_r(T') \leqslant |D| - 2 = n$. However T' has order $3n$, and so $\gamma_r(T') \geqslant n+1$, producing a contradiction. Hence v_3 is adjacent to only one leaf, ℓ_1 say. If now $v_{m-1} \in D$, then $D - \{\ell_1, v_m\}$ is a restrained dominating set of $T' = T - \{\ell, v_m\}$, producing a contradiction. Hence $v_{m-1} \notin D$, and therefore $v_{m-2} \notin D$. As with v_4, we may assume that $v_{m-3} \notin D$. By Fact 6 (with v_3 replaced by v_{m-2}), we may assume that v_{m-2} is a remote vertex of T. Then, as with v_3, v_{m-2} is adjacent to only one leaf, ℓ_2 say. If both v_3 and v_{m-2} have degree 3, then $D - \{\ell_1, \ell_2\}$ is a restrained dominating set of $T' = T - \{\ell_1, \ell_2\}$, producing a contradiction. Hence, we may assume that $\deg v_3 \geqslant 4$.

Let u be a neighbor of v_3 distinct from v_2, v_4, and ℓ_1. If $u \in D$, then we arrive at a contradiction as shown in Fact 6. Hence $u \notin D$. But then $D - \{v_1\}$ is a restrained dominating set of $T' = T - \{v_1, v_2\}$, whence, as shown earlier, $T' \in \mathscr{T}_{3n}$ and $T \in \mathscr{C}_{3n+2}$. $\square$

Theorem 2 now follows from Lemmas 7, 8, 10 and 11.

References

[1] G. Chartrand, L. Lesniak, Graphs & Digraphs: Third Edition, Chapman & Hall, London, 1996.
[2] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar, L.R. Markus, Restrained domination in graphs, Discrete Math. 203 (1999) 61–69.
[3] G.S. Domke, J.H. Hattingh, R.C. Laskar, L.R. Markus, A constructive characterization of trees with equal domination and restrained domination numbers, submitted for publication.

[4] G.S. Domke, J.H. Hattingh, M.A. Henning, L.R. Markus, Restrained domination in graphs with minimum degree two, submitted for publication.
[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1997.
[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Ed.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1997.
[7] M.A. Henning, Graphs with large restrained domination number, Discrete Math. 197/198 (1999) 415–429.
[8] J.A. Telle, A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees, SIAM J. Discrete Math. 10 (1997) 529–550.